Problem 1. Mixed Nash equilibria, admissibility of equilibria

Harry and Sally plan to go on a date but do not recall where they agreed to meet. It is way before cell phone time so they cannot communicate with either to check the plan either. Sally (player 1) prefers to go watch a Soccer match, whereas Harry prefers to go to see an Opera. They do prefer going to the same event over each of them going to their individual favorite event. Considering these, the payoff matrix is as follows:

$$\begin{array}{c} \\ \text{Soccer} \\ \text{Opera} \end{array} \begin{array}{c} \begin{array}{cc} \text{Soccer} & \text{Opera} \end{array} \\ \begin{bmatrix} (a, b) & (c, c) \\ (d, d) & (b, a) \end{bmatrix} \end{array}$$

a) Based on the above description, what is the relation between a, b, c, d?

b) What are the pure strategy Nash equilibria?

c) Consider the case in which $a = 3, b = 2, c = 1, d = 0$. Compute the mixed strategy Nash equilibrium. What is the probability of players coordinating and attending the same event in the mixed equilibrium?

d) Which of the equilibria computed in the previous part are admissible?

Bonus: read about correlated equilibrium, for example, in Section 2.2 of Fudenberg & Tirole. Discuss how correlation can ensure a payoff higher than any of the above Nash equilibria.

Solution:

a) Each individual preferring their own event gives us $a > b$. Further, we have $a > d$ and $a > c$ from the fact that both Harry and Sally would prefer being together than going to separate events. $b > c$ arises from the players' preferences of being with each other over going to their individual favorite event and $c > d$ arises from the players' preference of events if they don't go to the same one. Thus, we have $a > b > c > d$.

b) The pure Nash Equilibria are (soccer, soccer) and (opera, opera). Note the payoffs $J1$(soccer, soccer)= a, $J2$(soccer, soccer)= b. If Sally deviates, her payoff becomes $J1$(opera, soccer)= $d < a$. Thus, she cannot increase her payoff through unilateral deviation. Similarly, if Harry deviates, his payoff becomes $J2$(soccer, opera)= $c < b$. Thus, he cannot increase his payoff through unilateral deviation. We can similarly show that (opera, opera) is a pure Nash Equilibrium, and that no other pure Nash Equilibria exist.

c) We use the result from class on computing completely mixed Nash Equilibria:

$$Az^* = p^* 1 \qquad 1^\top z^* = 1 \tag{0.1}$$

$$y^{*\top} B = q^* 1 \qquad 1^\top y^* = 1. \tag{0.2}$$

Letting $z^* = \left[z_1^*, 1 - z_1^*\right]^\top$ and $y^* = \left[y_1^*, 1 - y_1^*\right]^\top$, (0.1) gives us $z_1^* = \frac{1}{4}$. Similarly, (0.2) give us $y_1^* = \frac{3}{4}$. Substituting yields $p^* = q^* = \frac{3}{2}$. Hence, a mixed Nash Equilibrium for the problem would have Sally attending the soccer game with probability $\frac{3}{4}$ and Harry attending the soccer game with probability $\frac{1}{4}$. This tells us the probability of both Harry and Sally attending soccer is $\frac{1}{4} * \frac{3}{4} = \frac{3}{16}$ and the probability of both Harry and Sally attending opera is $\frac{3}{4} * \frac{1}{4} = \frac{3}{16}$. Thus, they would coordinate $\frac{3}{16} + \frac{3}{16} = \frac{3}{8} < \frac{1}{2}$ of the time.

d) Only the pure Nash Equilbria are admissible, since neither $(3, 2)$ or $(2, 3)$ is better than the other. The mixed Nash Equilibria is not admissible, since $(3, 2) \succ (\frac{3}{2}, \frac{3}{2})$ and $(2, 3) \succ (\frac{3}{2}, \frac{3}{2})$.

Problem 2. Football game

Two football teams, called Team R and Team C, will soon play a match against each other. A football fan wants to use Game Theory to guess the strategy that the two teams will use at the beginning of the game. Both teams have used offensive (O), balanced (B), and defensive (D) strategies in recent games. The fan estimates that, depending on the initial strategy used, the goal difference in favor of Team R will be as follows:

$$\begin{array}{c} \text{C (minimizer)} \\ \text{R (maximizer)} \begin{array}{c} {\scriptstyle O} \\ {\scriptstyle B} \\ {\scriptstyle D} \end{array} \overset{\begin{array}{ccc} {\scriptstyle O} & {\scriptstyle B} & {\scriptstyle D} \end{array}}{\begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & -2 \\ 1 & 3 & -1 \end{bmatrix}} \end{array}$$

a) Dominant strategies: Does the football game problem have a dominant strategy equilibrium? If so, determine it. Otherwise, explain why it does not exist.

Solution: The dominant strategy of team C is D. Then, the dominant strategy of team R is O. Thus, the dominant strategy equilibrium of the football game is (O, D).

b) Pure strategies: Does the football game problem have a saddle-point equilibrium in pure strategies? If so, determine it. Otherwise, explain why it does not exist.

Solution: The security strategy of player R is:

$$\begin{aligned} \arg \max_{i \in \{O,B,D\}} \min_{j \in \{O,B,D\}} a_{ij} &= \arg \max_{i \in \{O,B,D\}} \left(\min_{j \in \{O,B,D\}} (2,2,2), \min_{j \in \{O,B,D\}} (3,4,-2), \min_{j \in \{O,B,D\}} (1,3,-1) \right) \\ &= \arg \max_{i \in \{O,B,D\}} (2,-2,-1) = O \end{aligned}$$

The security strategy of player C is:

$$\begin{aligned} \arg \min_{j \in \{O,B,D\}} \max_{i \in \{O,B,D\}} a_{ij} &= \arg \min_{j \in \{O,B,D\}} \left(\max_{i \in \{O,B,D\}} (2,3,1), \max_{i \in \{O,B,D\}} (2,4,3), \max_{j \in \{O,B,D\}} (2,-2,-1) \right) \\ &= \arg \min_{i \in \{O,B,D\}} (3,4,2) = D \end{aligned}$$

By Slide 34 of Lecture 2 (O, D) is the only security strategy, and thus (O, D) is a saddle-point equilibrium (Nash equilibrium). All saddle-point equilibria of a zero-sum game have the same value V^*, in this game

$$V^* = \underline{V} = \overline{V} = 2.$$

The only other outcomes with value 2 are (O, O) and (O, B) but they are not saddle-point equilibria because in both cases team R has an incentive to switch its strategy.

c) Mixed strategies:

- What mixed strategy does Team R need to play so that the outcome of the game becomes independent of the strategy/ies played by Team C?
  Solution: Team R needs to play $y = (1, 0, 0) \in \mathcal{Y}$ to ensure that the outcome of the game becomes independent of the strategies played by Team C.

- Find all saddle-point equilibria (pure and in mixed strategies).
  Solution: From *b*) we know that (O, D) is the unique pure saddle-point equilibrium. Furthermore, by Bauer's maximum principle all mixed saddle-point equilibria are a convex combination of the pure saddle point equilibria. Since there only exists a unique pure saddle-point equilibrium is follows that there are no other mixed saddle point equilibria.

Problem 3. Saddle point equilibria in zero-sum games

Consider a zero-sum game with cost matrix $A \in \mathbb{R}^{m \times n}$. Let $\underline{V}_m$ and $\overline{V}_m$ denote the mixed security strategies for player 2 (maximizer) and player 1 (minimizer) respectively.

a) Prove the following statement provided in the lecture notes:
A zero-sum game has a mixed saddle-point equilibrium if and only if

$$\underline{V}_m = \max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} y^\top Az = \min_{y\in\mathcal{Y}} \max_{z\in\mathcal{Z}} y^\top Az = \overline{V}_m$$

Hint: we proved the analogous result for pure strategies during the lecture.

b) In class, we derived the linear program corresponding to player 1, the minimizer. Using the same approach as the lecture notes, derive the linear program for finding the mixed Nash equilibrium for player 2, the maximizer.
Remark: Read about duality in optimization and in particular, in linear programming. You can show that the linear programs for the minimizer and maximizer are dual linear programs.

Solution:

a) Suppose (y^*, z^*) is a saddle point. Then we have:

$$(y^*)^\top Az \leq (y^*)^\top Az^* \leq y^\top Az^*$$

for all $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. By definition of the maximum, we have:

$$(y^*)^\top Az^* = \min_{y\in\mathcal{Y}} y^\top Az^* \leq \max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} y^\top Az = \underline{V}_m$$

Similarly, by the definition of the minimum we have:

$$(y^*)^\top Az^* = \max_{z\in\mathcal{Z}} (y^*)^\top Az \geq \min_{y\in\mathcal{Y}} \max_{z\in\mathcal{Z}} y^\top Az = \overline{V}_m$$

Hence, $\overline{V}_m \leq \underline{V}_m$. By the Min-Max property for general functions, we have that $\underline{V}_m \leq \overline{V}_m$. Thus, we must have $\overline{V}_m = \underline{V}_m$.

Conversely, suppose that $\overline{V}_m = \underline{V}_m$. Then

$$\underline{V}_m = \max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} y^\top Az = \min_{y\in\mathcal{Y}} y^\top Az^*,$$

where z^* maximizes $\min_{y\in\mathcal{Y}} y^\top Az$. Similarly

$$\overline{V}_m = \min_{y\in\mathcal{Y}} \max_{z\in\mathcal{Z}} y^\top Az = \max_{z\in\mathcal{Z}} (y^*)^\top Az,$$

where y^* minimizes $\max_{z\in\mathcal{Z}} y^\top Az$. By definition of the maximum and the minimum we must have that

$$\min_{y\in\mathcal{Y}} y^\top Az^* \leq (y^*)^\top Az^* \leq \max_{z\in\mathcal{Z}} (y^*)^\top Az.$$

Since the expression on the left is $\underline{V}_m$ and the one on the right is $\overline{V}_m$ which are equal by hypothesis, we have

$$\min_{y\in\mathcal{Y}} y^\top Az^* = (y^*)^\top Az^* = \max_{z\in\mathcal{Z}} (y^*)^\top Az$$

and so

$$(y^*)^\top Az \leq \max_{z\in\mathcal{Z}} (y^*)^\top Az = (y^*)^\top Az^* = \min_{y\in\mathcal{Y}} y^\top Az^* \leq y^\top Az^*$$

for all $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Hence, (y^*, z^*) is a saddle point point equilibrium.

b) Consider $\underline{V}_m$. We have

$$\begin{aligned}\underline{V}_m &= \max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} \sum_{i=1}^{m} \sum_{j=1^n} y_i z_j a_{ij} \\ &= \max_{z\in\mathcal{Z}} \min_{i} \sum_{j=1^n} z_j a_{ij},\end{aligned}$$

where the last equality follows from the result that the optimum over a simplex is at one of its vertices. This is equivalent to the following linear program:

$$\begin{aligned} &\max_{z,V_m} V_m \\ &\text{subject to } Az \geq \mathbf{1}V_m \\ &\qquad z \in \mathcal{Z},\ V_m \in \mathbb{R}. \end{aligned}$$

where $\mathbf{1}$ is the column vectors of ones.

Additional problems: Solve Exercises 3 and 4 in the slides 01-Static games.pdf, and Exercise 1 in the slides 02-Zero-sum games.

Solution of Exercise 3 (Properties of mixed strategies and payoffs)

a) A set $\mathcal{X} \subset \mathbb{R}^n$ is convex if for any $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$ the following holds:

$$\alpha x + (1-\alpha)y \in \mathcal{X}.$$

b) The set of mixed strategies $\mathcal{Y}$ is defined as:

$$\mathcal{Y} = \left\{ y \middle| \sum_{i=1}^{n} y_i = 1, \quad y_i \geq 0 \right\}.$$

Suppose any $x, y \in \mathcal{Y}$ and $\alpha \in [0, 1]$. First, we verify:

$$\begin{aligned} \sum_{i=1}^{n} \alpha x_i + (1-\alpha)y_i &= \alpha \sum_{i=1}^{n} x_i + (1-\alpha)\sum_{i=1}^{n} y_i \\ &= \alpha + (1-\alpha) \\ &= 1. \end{aligned}$$

Furthermore, for $i = 1, \dots, n$, it holds that $\alpha x_i + (1-\alpha)y_i \geq 0$ since $\alpha, (1-\alpha), x_i, y_i \geq 0$. Thus, $\mathcal{X}$ is a convex set.

c) Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a convex set. A function $f : \mathcal{X} \to \mathbb{R}$ is convex if for any $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$ the following holds:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

d) Assume z is fixed and define $x = Az$ The function $J_1(y, z) = y^\top Az = y^\top x$ is linear in y by definition of linear functions (see below for a reminder). Linear functions are convex (you should verify this) and thus $J_i(y, z)$ is a convex function in y when z is fixed.

Recall, given two linear spaces U, V over the field $\mathbb{R}$, a function $f : U \to V$ is linear if

$$\forall u_1, u_2 \in U, \alpha_1, \alpha_2 \in R,\ f(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 f(u_1) + \alpha_2 f(u_2).$$

Solution of Exercise 4 (Computing mixed Nash equilibria in matching pennies)

Consider the penalty kick:

$$\begin{array}{c|cc} & \text{Left} & \text{Right} \\ \text{Left} & (1,-1) & (-1,1) \\ \text{Right} & (-1,1) & (1,-1) \end{array}.$$

At the Nash equilibrium the optimal strategy $z^* = [z_1^*, 1-z_1^*]^\top$ has to respect the condition $Az^* = p^*1$, where

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We obtain the two following equations:

$$z_1^* - 1 + z_1^* = p^*,$$
$$-z_1^* + 1 - z_1^* = p^*.$$

The solution of the system is $p^* = 0$ and $z_1^* = \frac{1}{2}$. Thus the optimal strategy is $z^* = [\frac{1}{2}, \frac{1}{2}]^\top$. Due to the symmetry of the game the optimal strategy y^* is equal to z^*. Thus the mixed Nash equilibrium is $([\frac{1}{2}, \frac{1}{2}]^\top, [\frac{1}{2}, \frac{1}{2}]^\top)$. To double check, we can control that the conditions for being a mixed Nash equilibrium holds. The pair of mixed strategies $y^* \in \mathcal{Y}$ and $z^* \in \mathcal{Z}$ is a mixed Nash equilibrium if

$$(y^*)^\top A z^* \geq y^\top A z^*, \quad \forall y \in \mathcal{Y},$$

and if

$$(y^*)^\top B z^* \geq (y^*)^\top B z, \quad \forall z \in \mathcal{Z},$$

where

$$B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The mixed Nash equilibrium is

$$(y^*, \quad z^*) = \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right)$$

since

$$\begin{pmatrix} \frac{1}{2}, & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}, & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \geq \begin{pmatrix} y_1, & y_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall (y_1, y_2) \in \mathcal{Y}.$$

Analogous computations show that $(y^*)^\top B z^* \geq (y^*)^\top B z$ also holds for z^* and all $z \in \mathcal{Z}$.